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# Beauville $p$ -groups: a survey

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Beauville  $p$ -groups: a survey

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## Abstract

Beauville surfaces are a class of complex surfaces defined by letting a finite group  $G$  act on a product of Riemann surfaces. These surfaces possess many attractive geometric properties several of which are dictated by properties of the group  $G$ . In this survey we discuss the  $p$ -groups that may be used in this way. *En route* we discuss several open problems, questions and conjectures.

## 1 Introduction

Roughly speaking (precise definitions will be given in the next section), a Beauville surface is a complex surface  $\mathcal{S}$  defined by taking a pair of complex curves, i.e. Riemann surfaces,  $\mathcal{C}_1$  and  $\mathcal{C}_2$  and letting a finite group  $G$  act freely on their product to define  $\mathcal{S}$  as a quotient  $(\mathcal{C}_1 \times \mathcal{C}_2)/G$ . These surfaces have a wide variety of attractive geometric properties: they are surfaces of general type; their automorphism groups [42] and fundamental groups [6, 18] are relatively easy to compute (being closely related to  $G$ ); these surfaces are rigid surfaces in the sense of admitting no nontrivial deformations [8] and thus correspond to isolated points in the moduli space of surfaces of general type [19, 30].

Much of this good behaviour stems from the fact that the surface  $(\mathcal{C}_1 \times \mathcal{C}_2)/G$  is uniquely determined by a particular pair of generating sets of  $G$  known as a ‘Beauville structure’. This converts the study of Beauville surfaces to the study of groups with Beauville structures, i.e. Beauville groups.

Beauville surfaces were first defined by Catanese in [18] as a generalisation of an earlier example of Beauville [12, Exercise X.13(4)] (native English speakers may find the English translation [13] somewhat easier to read and get hold of) in which  $\mathcal{C} = \mathcal{C}'$  and the curves are both the Fermat curve defined by the equation  $X^5 + Y^5 + Z^5 = 0$  being acted on by the 5-group  $C_5 \times C_5$  (we write  $C_n$  for the cyclic group of order  $n$ . This choice of group may seem somewhat odd at first, but the reason will become clear later). Bauer, Catanese and Grunewald went on to use these surfaces to construct examples of smooth regular surfaces with vanishing geometric genus [9]. Early motivation came from the consideration of the ‘Friedman-Morgan speculation’ — a technical conjecture concerning when two algebraic surfaces are diffeomorphic which Beauville surfaces provide counterexamples to. More recently, they have been used to construct interesting orbits of the absolute Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  [32, 35, 36] (connections with Gothenick’s theory of dessins d’enfant make it possible for this group to act on the set of all

Beauville surfaces). Furthermore, Beauville's original example has also been used by Galkin and Shinder in [29] to construct examples of exceptional collections of line bundles.

Whilst these constructions work for finite groups in general there is particular interest in studying the special case of  $p$ -groups. First note that in some sense 'most finite groups are  $p$ -groups': there are 49 910 529 484 groups of order at most 2 000. Of these groups 49 487 365 422 have order precisely 1 024 — that is more than  $99 \cdot 1\%$  of the total! When we add to this collection all the other  $p$ -groups of order at most 2 000 we have essentially all of them. By comparison the total number of finite simple groups of order at most 2000 is merely six! Answering questions about Beauville groups in the case of  $p$ -groups thus goes a long way to answering these questions for finite groups in general. (For details of the extraordinary and impressive computational feats just mentioned and a general historical discussion of the problem of enumerating groups of small order, which has been worked on for almost a century and a half, see the work of Besche, Eick and O'Brien in [15, 16].) Moreover, as we shall see later, there are many reasons for believing that Beauville  $p$ -groups with various properties are much harder to construct compared to other cases.

Like any survey article, the topics discussed here reflect the research interests of the author. In many ways this survey is the sequel to authors contribution to the last Groups St Andrews proceedings [22] though the reader will lose little if have neither read nor have to hand a copy of [22]. Slightly older surveys discussing related geometric and topological matters are given by Bauer, Catanese and Pignatelli in [10, 11]. Other notable works in the area include [5, 43, 51, 54].

This survey is organised as follows. In Section 2 we will give precise definitions for the various Beauville constructions described in vague terms above. In Section 3 we will describe general constructions of Beauville  $p$ -groups before moving on in Section 4 to focus on the so-called 'mixed case' and in Section 5 we will focus on strongly real examples. Finally, in Section 6 we discuss open questions, problems and conjectures in the area.

## 2 Main Definitions

**Definition 2.1** A surface  $\mathcal{S}$  is a **Beauville surface of unmixed type** if

- the surface  $\mathcal{S}$  is isogenous to a higher product, that is,  $\mathcal{S} \cong (\mathcal{C}_1 \times \mathcal{C}_2)/G$  where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are algebraic curves of genus at least 2 and  $G$  is a finite group acting faithfully on  $\mathcal{C}_1$  and  $\mathcal{C}_2$  by holomorphic transformations in such a way that it acts freely on the product  $\mathcal{C}_1 \times \mathcal{C}_2$ , and
- each  $\mathcal{C}_i/G$  is isomorphic to the projective line  $\mathbb{P}_1(\mathbb{C})$  and the covering map  $\mathcal{C}_i \rightarrow \mathcal{C}_i/G$  is ramified over three points.

There also exists a concept of Beauville surfaces of mixed type but we shall postpone our discussion of these until Section 4. In the first of the above conditions the genus of the curves in question needs to be at least 2. It was later proved by Fuertes, González-Díez and Jaikin-Zapirain in [27] that in fact we can take

the genus as being at least 6. The second of the above conditions implies that each  $\mathcal{C}_i$  carries a regular dessin in the sense of Grothendieck's theory of *dessins d'enfants* (children's drawings) [37]. Furthermore, by Belyi's Theorem [14] this ensures that  $\mathcal{S}$  is defined over an algebraic number field in the sense that when we view each Riemann surface as being the zeros of some polynomial we find that the coefficients of that polynomial belong to some number field. Equivalently they admit an orientably regular hypermap [46], with  $G$  acting as the orientation-preserving automorphism group. A modern account of dessins d'enfants and proofs of Belyi's theorem may be found in the recent book of Gironde and González-Diez [31]. An even more recent account of dessins d'enfants, which culminates in a final chapter on Beauville surfaces, is given by Jones and Wolfart in [47].

These surfaces can also be described instead in terms of uniformisation and the language of Fuchsian groups [34, 53].

What makes this class of surfaces so good to work with is the fact that all of the above definition can be 'internalised' into the group. It turns out that a group  $G$  can be used to define a Beauville surface if and only if it has a certain pair of generating sets known as a Beauville structure.

**Definition 2.2** Let  $G$  be a finite group. Let  $x, y \in G$  and let

$$\Sigma(x, y) := \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

An **unmixed Beauville structure** for the group  $G$  is a set of pairs of elements  $\{\{x_1, y_1\}, \{x_2, y_2\}\} \subset G \times G$  with the property that  $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$  such that

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{e\}.$$

If  $G$  has a Beauville structure we say that  $G$  is a **Beauville group**. Furthermore we say that the structure has **type**

$$((o(x_1), o(y_1), o(x_1y_1)), (o(x_2), o(y_2), o(x_2y_2))).$$

Traditionally, authors have defined the above structure in terms of so-called 'spherical systems of generators of length 3', meaning  $\{x, y, z\} \subset G$  with  $xyz = e$ , but we omit  $z = (xy)^{-1}$  from our notation in this survey. (The reader is warned that this terminology is a little misleading since the underlying geometry of Beauville surfaces is hyperbolic thanks to the below constraint on the orders of the elements.) Furthermore, many earlier papers on Beauville structures add the condition that for  $i = 1, 2$  we have that

$$\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_iy_i)} < 1,$$

but this condition was subsequently found to be unnecessary following Bauer, Catanese and Grunewald's investigation of the wall-paper groups in [7]. A triple of

elements and their orders satisfying this condition are said to be hyperbolic. Geometrically, the type gives us considerable amounts of geometric information about the surface: the Riemann-Hurwitz formula

$$g(\mathcal{C}_i) = 1 + \frac{|G|}{2} \left( 1 - \frac{1}{o(x_i)} - \frac{1}{o(y_i)} - \frac{1}{o(x_i y_i)} \right)$$

tells us the genus of each of the curves used to define the surface  $\mathcal{S}$  and by a theorem of Zeuthen-Segre this also gives us the Euler number of the surface  $\mathcal{S}$  since

$$e(\mathcal{S}) = 4 \frac{(g(\mathcal{C}_1) - 1)(g(\mathcal{C}_2) - 1)}{|G|}$$

which in turn gives us the holomorphic Euler-Poincaré characteristic of  $\mathcal{S}$ , namely  $4\chi(\mathcal{S}) = e(\mathcal{S})$  (see [18, Theorem 3.4]).

Already we see one reason for finding the  $p$ -groups case more challenging — in many places in the literature a commonly used trick to show that the condition that  $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{e\}$  is satisfied is to find a Beauville structure such that  $o(x_1)o(y_1)o(x_1 y_1)$  is coprime to  $o(x_2)o(y_2)o(x_2 y_2)$  but this clearly cannot be done in a  $p$ -group since every non-trivial element has an order that is a power of  $p$ .

There are several properties a  $p$ -group may have that make it intuitively more likely to be a Beauville group. Having a low exponent makes it easier for  $\Sigma(g, h)$  to be small, indeed it is not difficult to prove that if  $n > 2$  then a Beauville  $p$ -group of order  $p^n$  must have exponent at most  $p^{n-2}$  (although there do exist Beauville  $p$ -groups attaining this bound). Moreover have a large abelian subgroup, and in particular a large center, makes it easier for elements to have large centralizers and thus belong to small conjugacy classes, again making  $\Sigma(g, h)$  small.

### 3 General Constructions

The earliest examples of Beauville  $p$ -groups were given in Catanese's original paper [18] where he showed that the groups  $C_n \times C_n$  are Beauville groups whenever  $n > 1$  is coprime to 6. (Later in [7] Bauer, Catanese and Grunewald showed that these are in fact the only abelian Beauville groups. A proof adapted from theirs is given by Jones and Wolfart in [47, Theorem 11.1].) In particular if  $p > 3$  is a prime, then we can take  $n$  to be a power of  $p$  giving infinitely many examples of (abelian) Beauville  $p$ -groups, though alas this also tells us that there are no abelian examples at all when  $p = 2$  or  $3$ . This explains Beauville's original choice for his example —  $C_5 \times C_5$  is the smallest abelian group that is a Beauville group. Subsequently in [33] Gonzalez-Diez, Jones and Torres-Teigell put this classification to great use deriving a number of interesting facts about the surfaces associated with these groups. For example they showed that the field of definition of the corresponding surfaces is always  $\mathbb{Q}$ . They also showed that if  $q$  is a power of a prime  $p > 3$ , then the number of isomorphism classes of Beauville surfaces coming from the Beauville group  $C_q \times C_q$  is asymptotically  $q^4/72$ .

The earliest examples of Beauville 2-groups and 3-groups (that are unmixed — we postpone a brief discussion of some slightly older mixed Beauville 2-groups until

the next section) were given by Fuertes, González-Diez and Jaikin-Zapirain in [27, Section 5] where isolated examples of Beauville groups of order  $2^{12}$  and  $3^{12}$  are constructed.

The earliest systematic attempt to construct infinite families of Beauville  $p$ -groups was by Barker, Boston and the author in [1] where they considered groups of order  $p^n$  for small values of  $n$ . In particular they found the smallest Beauville  $p$ -group for every  $p$  by proving the following [1, Corollary 9].

**Lemma 3.1** *The smallest non-abelian Beauville  $p$ -groups are as follows:*

(a) *for  $p = 2$  the group of order  $2^7$  defined by the presentation*

$$\langle x, y \mid x^4, y^4, [x^2, y^2], (y^3 x^3 y^3 x)^2, (x^3 y^3 x y)^2, (y^3 x y x)^2, (y^3 x^2)^2 y x^2 y x^2, (x^2 y^3)^4 \rangle$$

(b) *for  $p = 3$  the group of order  $3^5$  defined by presentation*

$$\langle x, y, z, w, t \mid x^3, y^3, z^3, w^3, t^3, y^x = yz, z^x = zw, z^y = zt \rangle$$

*where whenever any two of the generators,  $g$  and  $h$ , commute we omitted the relation  $[g, h] = e$  for clarity and*

(c) *for  $p \geq 5$  the extraspecial group  $p_+^{1+2}$  defined by the presentation*

$$\langle x, y, z \mid x^p, y^p, z^p, yx = xyz, [x, z], [y, z] \rangle.$$

They went on to show that for every  $p \geq 5$  and  $n \geq 4$  there exists a non-Beauville group of order  $p^n$  and there exists at least one Beauville group of order  $p^n$ . They then went on to completely classify the Beauville groups of order at most  $p^4$  and go most of the way to classifying the Beauville groups of order  $p^5$  and  $p^6$  (more on this later). One interesting feature of this work was the observation that as  $p \rightarrow \infty$  the proportion of 2-generated groups of order  $p^5$  that are Beauville tends to 1, however as  $p \rightarrow \infty$  the proportion of 2-generated groups of order  $p^6$  that are Beauville does not tend to 1!

In [4] the Barker, Boston, Peyerimhoff and Vdovina, for reasons that will become slightly clearer in the next section, considered quotients of the group

$$\langle x_0, x_1, \dots, x_{12} \mid x_i^3, x_i x_{i+1} x_{i+4} \text{ for } i = 0, \dots, 13 \rangle$$

(the indices in the relations should be read modulo 13) and the index 3 subgroup generated by  $x_0, x_1$  and  $x_2$ . Doing this they were able to construct a finite number of Beauville 3-groups, conjecturing that an infinite family can be obtained this way.

As an interesting application of these ideas in [49] Peyerimhoff and Vdovina were able to obtain expander graphs (infinite families of finite graphs that are sparse and yet simultaneously highly connected, which enjoy applications in computer science for example in the design of highly robust networks) from Cayley graph corresponding to generating sets for these groups.

More recently in [25] Fernández-Alcober and Gül took a rather different approach. First they prove the following general result for constructing Beauville  $p$ -groups when  $p \geq 5$ .

**Theorem 3.2** *Let  $G$  be a 2-generator finite  $p$ -group of exponent  $p^e$  and suppose that  $G$  satisfies one of the following conditions*

- (a) *For  $x, y \in G$ , we have  $x^{p^{e-1}} = y^{p^{e-1}}$  if and only if  $(xy^{-1})^{p^{e-1}} = e$ .*
- (b)  *$G$  is a potent  $p$ -group.*

*Then  $G$  is a Beauville group if and only if  $p \geq 5$  and  $|G^{p^{e-1}}| \geq p^2$ . If that is the case, then every lift of a Beauville structure of  $G/\Phi(G)$  is a Beauville structure of  $G$ .*

Here a finite  $p$ -group is said to be ‘potent’ if either  $p > 2$  and  $\gamma_{p-1}(G) \leq G^p$  or  $p = 2$  and  $G' \leq G^4$  where  $\gamma_i(G)$  is inductively defined by the lower central series  $\gamma_1(G) = G$  and for  $i > 1$  we have that  $\gamma_i(G) = [\gamma_{i-1}(G), G]$ .

In later parts of [25] Fernández-Alcober and Gül consider quotients of the famous Nottingham group to construct more infinite families of  $p$ -groups for  $p$  odd and in particular gave the first infinite family of Beauville 3-groups.

In [17] Boston discussed Beauville  $p$ -groups in terms of the ‘O’Brien tree’ of  $p$ -groups. Given a 2-generated  $p$ -group with lower  $p$ -central series of subgroups  $(P_i(G))_{i=1}^n$  (recall that this is defined inductive by  $P_0(G) = G$  and for  $i > 0$ ,  $P_{i+1}(G) = [G, P_i(G)]P_i(G)^p$ ) we have a sequence of quotients

$$G = G/P_n(G) \rightarrow G/P_{n-1}(G) \rightarrow \cdots \rightarrow G/P_1(G) = C_p^2.$$

For a fixed  $p$  we define a tree whose vertices are the isomorphism classes of 2-generated  $p$ -groups (the most general construction drops the 2-generated restriction, but in this setting it seems appropriate to make this assumption). A  $p$ -group  $G$  of class  $i$  is then adjoined to any  $p$ -group  $H$  of class  $i+1$  such that  $H/P_i(H) \cong G$  (these are called the ‘children’ of  $G$ ). Boston investigated the relationship between a  $p$ -group’s status as a Beauville group and the status of its children surprisingly finding little correlation between the two and even finding  $p$ -groups with infinitely many children none of which are Beauville. There does, however appear to be some connection with a quantity associated with the O’Brien tree known as the ‘nuclear rank’ of a  $p$ -group. See [48] for the technical details.

The following general lemma of Fuertes and Jones [28, Lemma 4.2] (originally proved with special linear groups in mind) has been applied in various parts of the literature to Beauville  $p$ -groups.

**Lemma 3.3** *If  $x, y, u, v \in G$  have images  $\bar{x}, \bar{y}, \bar{u}, \bar{v} \in G/N$  yielding a Beauville structure in  $G/N$ , then if*

$$\langle x \rangle \cap N = \langle y \rangle \cap N = \langle z \rangle \cap N = \{e\},$$

*then  $x, y, u, v$  give a Beauville structure in  $G$ .*

In particular in [17, Corollary 3.3] deduces the following.

**Corollary 3.4** *If  $p \geq 5$  and  $\Gamma$  is the triangle group*

$$\langle x, y, z \mid x^p = y^p = z^p = xyz = e \rangle,$$

*then its  $p$ -central quotients  $\Gamma/P_i(\Gamma)$  are all Beauville groups.*



On the basis of this Boston conjectured that if the group  $\Gamma$  is either the free product  $\langle x, y \mid x^p, y^p \rangle$  or the free group of rank 2, then  $\Gamma/P_i(\Gamma)$  are all Beauville. Both cases of this conjecture were later proved in the affirmative in [38] by Gül who also noted that when comparing the infinite family of Beauville 3-groups this gives with the family discussed above that were constructed in [25] the only group lying in both families was the smallest one, namely the Beauville group of order  $3^5$  as given in Lemma 3.1.

In [52] Stix and Vdovina took a more global view of all Beauville  $p$ -groups by deploying the theory of pro- $p$  groups (leading to the introduction of a new notion, that of a ‘topological Beauville structure’) to prove the following.

**Theorem 3.5** *Every finite  $p$ -group with an unmixed Beauville structure sits in an infinite pro-system of  $p$ -groups with compatible unmixed Beauville structure such that the type of the first half remains constant throughout the pro-system.*

In the same paper they were able to obtain a new infinite family of non-abelian Beauville  $p$ -groups by proving the following.

**Theorem 3.6** *Let  $m, n \in \mathbb{N}$  and  $\lambda \in (\mathbb{Z}/p^m\mathbb{Z})^\times$  with  $\lambda^{p^n} \equiv 1 \pmod{p^m}$ . The semidirect product*

$$\mathbb{Z}/p^m\mathbb{Z} :_{\lambda} \mathbb{Z}/p^n\mathbb{Z}$$

*with action  $\mathbb{Z}/p^n\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/p^m\mathbb{Z})$  sending  $1 \mapsto \lambda$  admits an unmixed Beauville structure if and only if  $p \geq 5$  and  $m = n$ .*

One of the most recent constructions of Beauville  $p$ -groups comes from the following general criterion given by Jones and Wolfart in [47, Chapter 11].

**Theorem 3.7** *Let  $G$  be a finite group of exponent  $n = p^e > 1$  for some prime  $p \geq 5$ , such that the abelianisation  $G/G'$  of  $G$  is isomorphic to  $C_p \times C_p$ . Then  $G$  is a Beauville group.*

**Corollary 3.8** *Let  $G$  be a 2-generated finite group of exponent  $p$  for some prime  $p \geq 5$ . Then  $G$  is a Beauville group.*

To give concrete examples of groups satisfying the hypotheses of these results they leave to the reader the exercise of showing that if  $W$  is the wreath product  $C_n \wr C_n$ , then the quotient of this group by the ‘diagonal subgroup’ (i.e. the center) is a group that satisfies the hypotheses of Theorem 3.7 though they also remark that “Since  $p$ -groups tend to have many quotients, these results show that there is no shortage of groups satisfying the hypotheses of [these results].”

Here we have discussed general constructions of Beauville  $p$ -groups that (at the time of writing) are not known to have any particular additional properties from the viewpoint of being Beauville groups. In the next two sections we discuss constructions of Beauville  $p$ -groups that are known to have additional properties, namely the properties of being ‘mixed’ and the property of being ‘strongly real’.

## 4 The Mixed Case

When we defined Beauville surfaces and groups we considered the action of a group  $G$  on the product of two curves  $\mathcal{C}_1 \times \mathcal{C}_2$ . In an unmixed structure this action comes solely from the action of  $G$  on each curve individually, however there is nothing to stop us considering an action on the product that interchanges the two curves and it is precisely this situation that we discuss in this section. Recall from Definition 2.2 that given  $x, y \in G$  we write

$$\Sigma(x, y) := \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

**Definition 4.1** Let  $G$  be a finite group. A **mixed Beauville structure** for  $G$  is a quadruple  $(G^0, g, h, k)$  where  $G^0$  is an index 2 subgroup and  $g, h, k \in G$  are such that

- $\langle g, h \rangle = G^0$ ;
- $k \notin G^0$ ;
- for every  $\gamma \in G^0$  we have that  $(k\gamma)^2 \notin \Sigma(g, h)$  and
- $\Sigma(g, h) \cap \Sigma(g^k, h^k) = \{e\}$

A Beauville surface defined by a mixed Beauville structure is called a **mixed Beauville surface** and a group possessing a mixed Beauville structure is called a **mixed Beauville group**.

In terms of the curves defining the surface, the group  $G^0$  is the stabiliser of the curves with the elements of  $G \setminus G^0$  interchanging the two terms of  $\mathcal{C}_1 \times \mathcal{C}_2$ . Moreover it is only possible for a Beauville surface  $(\mathcal{C}_1 \times \mathcal{C}_2)/G$  to come from a mixed Beauville structure if  $\mathcal{C}_1 \cong \mathcal{C}_2$ . The above conditions also ensure that  $\{\{g, h\}, \{g^k, h^k\}\} \subset G^0 \times G^0$  is a Beauville structure for  $G^0$ .

In general, mixed Beauville structures are much harder to construct than their unmixed counterparts and, as noted in the introduction, it is even harder still in the case of  $p$ -groups: since a mixed Beauville group necessarily has an index 2 subgroup we must have  $p = 2$ . In particular, this fact combined with Catanese's classification of the abelian Beauville groups that we discussed at the beginning of Section 3 we can immediately see that there are no abelian mixed Beauville groups.

The following lemma of Fuertes and González-Díez imposes a strong condition on a group with a mixed Beauville structure [26, Lemma 5].

**Lemma 4.2** *Let  $(\mathcal{C}_1 \times \mathcal{C}_2)/G$  be a mixed Beauville surface and let  $G^0$  be the subgroup of  $G$  consisting of the elements which do not interchange the two curves. Then the order of any element in  $G \setminus G^0$  is divisible by 4.*

Some of the earliest examples of Beauville 2-groups were constructed in an effort to find examples of mixed Beauville groups. In [9] Bauer, Catanese and Grunewald reported computer calculations they had done to find small examples verifying that there were none of order strictly less than  $2^8$  and only two such groups of order

$2^8$  which have the same index 2 subgroup. Explicitly the index 2 subgroup is the one mentioned in part (a) of Lemma 3.1. The groups themselves are given by the presentations

$$\begin{aligned} \langle x_1, \dots, x_8 \mid & x_1^2 = x_4x_5x_6, x_2^2 = x_4x_5, x_3^2 = x_4, x_4^2 = x_5^2 = x_6^2 = x_7^2 = x_8^2 = e \\ & x_2^{x_1} = x_2x_4, x_3^{x_1} = x_3x_5, x_3^{x_2} = x_3x_6, \\ & x_4^{x_1} = x_4x_7, x_4^{x_2} = x_4x_8, x_5^{x_1} = x_5x_7x_8, \\ & x_5^{x_2} = x_5x_8, x_5^{x_3} = x_5x_7, x_6^{x_1} = x_6x_8, \\ & x_6^{x_2} = x_6x_7, x_6^{x_3} = x_6x_8 \rangle \end{aligned}$$

and

$$\begin{aligned} \langle x_1, \dots, x_8 \mid & x_1^2 = x_4x_5x_6x_7, x_2^2 = x_4x_5, x_3^2 = x_4, x_4^2 = x_5^2 = x_6^2 = x_7^2 = x_8^2 = e \\ & x_2^{x_1} = x_2x_4, x_3^{x_1} = x_3x_5, x_3^{x_2} = x_3x_6, \\ & x_4^{x_1} = x_4x_7, x_4^{x_2} = x_4x_8, x_5^{x_1} = x_5x_7x_8, \\ & x_5^{x_2} = x_5x_8, x_5^{x_3} = x_5x_7, x_6^{x_1} = x_6x_8, \\ & x_6^{x_2} = x_6x_7, x_6^{x_3} = x_6x_8 \rangle. \end{aligned}$$

the index 2 subgroup being the group defined in Lemma 3.1.

Six more examples of orders  $2^{14}$ ,  $2^{16}$ ,  $2^{19}$ ,  $2^{24}$ ,  $2^{27}$  were constructed by Barker, Boston, Peyerimhoff and Vdovina in [2] where they conjectured that their method could be adapted to give an infinite family of mixed Beauville 2-groups. They later verified this to be correct in [3] by constructing an infinite family of Beauville 2-groups as quotients of the group

$$\langle x_0, x_1, \dots, x_6 \mid x_i x_{i+1} x_{i+3} \text{ for } i = 0, \dots, 6 \rangle$$

(the indices should be read modulo 7) and the index 2 subgroup generated by  $x_0$  and  $x_1$  only. The relations in the above presentation should immediately make the reader think of the famous Fano plane and generalising in this direction is indeed a subsequent development of the subject it being just one example of a ‘group with special presentation’ defined this way. This gave the first infinite family of mixed Beauville 2-groups were constructed by Barker, Boston, Peyerimhoff and Vdovina in [3].

As far as the author is aware, the most recent general discussion of examples of mixed Beauville groups is given by the author and Pierro in [24] (though none of the Beauville groups constructed in [24] are even soluble, let alone are  $p$ -groups).

## 5 The Strongly Real Case

Given any complex surface  $\mathcal{S}$  it is natural to consider the complex conjugate surface  $\overline{\mathcal{S}}$ . In particular it is natural to ask if the surfaces are biholomorphic.

**Definition 5.1** Let  $\mathcal{S}$  be a complex surface. We say that  $\mathcal{S}$  is **real** if there exists a biholomorphism  $\sigma : \mathcal{S} \rightarrow \overline{\mathcal{S}}$  such that  $\sigma \circ \overline{\sigma}$  is the identity map.

As noted earlier this geometric condition can be translated into algebraic terms.

**Definition 5.2** Let  $G$  be a Beauville group and let  $X = \{\{x_1, y_1\}, \{x_2, y_2\}\}$  be a Beauville structure for  $G$ . We say that  $G$  and  $X$  are **strongly real** if there exists an automorphism  $\phi \in \text{Aut}(G)$  and elements  $g_i \in G$  for  $i = 1, 2$  such that

$$g_i \phi(x_i) g_i^{-1} = x_i^{-1} \text{ and } g_i \phi(y_i) g_i^{-1} = y_i^{-1}.$$

It is often, but not always, convenient to take  $g_1 = g_2 = e$ .

Again what makes the case of  $p$ -groups particularly interesting is how much harder it is to construct strongly real examples in this case compared to groups in general, especially when  $p$  is odd. In [41] Helleloid and Martin prove that automorphism group of a finite  $p$ -group is almost always a  $p$ -group. In particular, if  $p$  is odd, then typically no automorphism like the  $\phi$  in of Definition 5.2 exists since such an automorphism must necessarily have even order. Nonetheless many examples have been found.

We first note that every abelian Beauville group is strongly real since the function  $x \mapsto -x$  is always an automorphism of an abelian group and in particular the abelian Beauville  $p$ -groups discussed at the beginning of Section 3 are strongly real.

As far as the author is aware the earliest examples of non-abelian strongly real Beauville  $p$ -groups to be discovered were an isolated pair of examples of 2-groups constructed by the author in [20, Section 7] namely the groups

$$\langle u, v \mid (u^i v^j)^4, i, j = 0, 1, 2, 3 \rangle$$

which has order  $2^{14}$  and

$$\langle u, v \mid u^8, v^8, [u^2, v^2], (u^i v^j)^4, i, j = 1, 2, 3 \rangle$$

which has order  $2^{13}$ . We take this opportunity to correct an error made by the author in the original proof of [20, Lemma 3] where this was first proved. In the original proof it is stated that we can take  $x_1 := u$ ,  $y_1 := v$ ,  $x_2 := uvu$  and  $y_2 := vuv$  as our generators however these elements clearly cannot provide a Beauville structure because  $x_2 y_2 \in \Sigma(x_1, y_1)$  since  $x_2 y_2 = (x_1 y_1)^3$ . If, however, we instead take  $y_2 := uvuvu$  then the function mapping  $u \leftrightarrow u^{-1}$  and  $v \leftrightarrow v^{-1}$  still inverts this new  $y_2$ . It is easy to see that  $\langle x_2, y_2 \rangle = G$  since  $vu = x_2^{-1} y_2$ ;  $v = x_2 (vu)^{-1}$  and  $u = v^{-1} (vu)$  hence  $u, v \in \langle x_2, y_2 \rangle$ . The other conditions of being a Beauville structure are also easily checked by computer.

Recently in [39] Gül constructed the first known infinite family of non-abelian strongly real Beauville  $p$ -groups and in particular discovered the first examples in which  $p$  is odd. More specifically, the main result of [39] is the following.

**Theorem 5.3** *Let  $F = \langle x, y \mid x^p, y^p \rangle$  be the free product of two cyclic groups of order  $p$  for an odd prime  $p$  and let  $i = k(p-1) + 1$  for  $k \geq 1$ . Then the quotient  $F/\gamma_{i+1}(F)$  is a strongly real Beauville group.*

Subsequently in [40] Gül constructed further examples by considering quotients of certain triangle groups. More specifically Gül prove that there are non-abelian

strongly real Beauville  $p$ -groups of order  $p^n$  for every  $n \geq 3$ , 5 or 7 for the primes  $p \geq 5$ ,  $p = 3$  and  $p = 7$  respectively.

At around the same time the author constructed another infinite family of non-abelian strongly real Beauville  $p$ -groups for  $p$  odd in [23] by proving the following.

**Theorem 5.4** *Let  $p$  be an odd prime and let  $q$  and  $r$  be powers of  $p$ . If  $q$  and  $r$  are sufficiently large, then groups  $C_q \wr C_r / Z(C_q \wr C_r)$  are strongly real Beauville groups.*

Unlike the groups given by Theorem 5.3 this theorem gives multiple non-isomorphic examples for infinitely many orders. For example when  $(q, r) = (3^{28}, 3^3)$  or  $(q, r) = (3^3, 3^5)$  we obtain groups of order  $3^{731}$  which cannot be isomorphic since they have centers of different orders. Moreover, the proof of Theorem 5.4 is really just a special case of the following much more general construction.

**Lemma 5.5** *Let  $G$  be a finite group; let  $Z \leq G$  be a characteristic subgroup; let  $t \in \text{Aut}(G)$  and let  $x_1, y_1, x_2, y_2 \in G$  have the properties that*

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) \subseteq Z,$$

$$\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G \text{ and}$$

$$x_i^t = x_i^{-1} \text{ and } y_i^t = y_i^{-1} \text{ for } i = 1, 2.$$

*Then  $G/Z$  is a strongly real Beauville group.*

As far as the author is aware the best references for what is known about strongly Beauville groups more generally are the surveys given by the author in [20, 21].

## 6 Open Questions, Problems and Conjectures

### 6.1 Field of Definition of Beauville Surfaces

In Section 3 we mentioned the results of [33] due to González-Diez, Jones and Torres-Teigell on surfaces defined by abelian Beauville  $p$ -groups with  $p \geq 5$ . Determining the field of definition of a Beauville surface is in general a difficult problem. The next most easy cases after abelian  $p$ -groups seem to be the smallest non-abelian Beauville  $p$ -groups as classified by Barker, Boston and the author in [1].

**Question 6.1** What are the fields of definition of the Beauville surfaces defined by the smallest Beauville  $p$ -groups?

A related question is the following.

**Question 6.2** Does being defined by a Beauville  $p$ -group have geometric consequences for the underlying surface and if so what are they?

## 6.2 Beauville Dimension

In unpublished correspondence with the author Gareth A. Jones has recently introduced the intriguing notion ‘Beauville dimension’ that we define as follows.

**Definition 6.3** Let  $G$  be a finite group. A group  $G$  is said to have *Beauville dimension*  $n$  if there exist  $n$  pairs of elements  $x_1, \dots, x_n, y_1, \dots, y_n \in G$  such that  $\langle x_i, y_i \rangle = G$  for every  $i = 1, \dots, n$ ;

$$\bigcap_{i=1}^n \Sigma(x_i, y_i) = \{e\}$$

and no set of  $n - 1$  pairs of generators can be found with this property. If no such  $n$  exists then we say that  $G$  has *infinite Beauville dimension*.

By way of examples from  $p$ -groups, clearly any Beauville group has Beauville dimension 2 whilst dihedral 2-groups all have infinite Beauville dimension since for any generating pair  $g$  and  $h$  the set  $\Sigma(g, h)$  will necessarily contain the central involution. (Another, albeit non-nilpotent, example is the alternating group  $A_5$ : in this case  $\Sigma(g, h)$  necessarily contains elements from the only class of cyclic subgroup of order 5.)

What makes this concept interesting is as follows. Having Beauville dimension  $n$  means that the group  $G$  can act on a product of  $n$  Riemann surfaces with many of the nice geometric properties enjoyed by Beauville surfaces that we discussed in Section 1 also being enjoyed by these higher dimensional manifolds and varieties (thanks to Serre’s GAGA principal they will be both varieties and manifolds). In short, this enables a higher dimensional theory of Beauville-like constructions.

As far as the author is aware the only known examples of groups with finite Beauville dimension greater than 2 are as follows. In personal correspondence Jones observed that the 3-groups  $C_{3^a} \times C_{3^a}$  all have Beauville dimension 4, an easy exercise for the reader. The only non-abelian examples the author has been able to find and is aware of are groups closely related to these examples such as the extraspecial group  $3_+^{1+2}$  and the wreath product  $C_3 \wr C_3$  which both have Beauville dimension 4.

All of this immediately poses the following question.

- Question 6.4** (a) Do there exist finite groups  $G$  with Beauville dimension 3?  
 (b) Do there exist groups of Beauville dimension  $n > 4$ ?  
 (c) Is the Beauville dimension of 2-generated finite groups bounded or can it get arbitrarily large?

Whilst this is not a question that specifically focuses on  $p$ -groups per se, it does seem that  $p$ -groups, in particular 3-groups, are a fertile breeding ground for examples that will address the above question. Further reason for believing that this is really a question concerning  $p$ -groups is the fact that the only finite quasi-simple groups that do not have Beauville dimension 2 are the alternating group  $A_5$  and its covering group  $SL_2(5)$  both of which have infinite Beauville dimension.

Characteristically simple groups similarly seem to be (and are conjectured to be — see [44, 45]) Beauville whenever they are 2-generated whilst something similar is true for almost simple groups. It follows that non-nilpotent examples will be very likely to be at best soluble.

### 6.3 What proportion of 2-generated $p$ -groups are Beauville groups?

As previously mentioned in Section 3 the question of what proportion of 2-generated  $p$ -groups are Beauville groups as raised by Barker, Boston and the author in [1] is particularly interesting since it seems that ‘most’ groups of order  $p^5$  are Beauville whereas we cannot say the same things about groups of order  $p^6$ . It is natural to ask which of these two situations are typical of groups of order  $p^n$  for general  $n$ .

**Question 6.5** For which  $n$  does the proportion of 2-generated groups of order  $p^n$  that are Beauville tend to 1? When it does not tend to 1 does it tend to 0?

One need only look at the number of groups of order at most  $p^5$  to realise this is the point where the number of groups of order  $p^n$  suddenly starts depending on  $p$  and it just happens that  $n = 5$  is sufficiently tame for the proportion to tend to 1, however as  $n$  increases the formula for the number of groups of order  $p^n$  becomes even more intertwined with the value of  $p$ . In the opinion of the author it is unlikely that the proportion of 2-generated groups of order  $p^n$  that are Beauville groups will tend to 1 as  $p$  tends to infinity for any  $n > 5$ .

### 6.4 Are Beauville groups typically strongly real?

As mentioned in Section 1 one motivation for focusing on the  $p$ -groups case is that in some sense ‘most’ finite groups are  $p$ -groups. It follows that the general question of what proportion of 2-generated  $p$ -groups are strongly real Beauville groups naturally translates to one of focusing on  $p$ -groups. The following is posed by the Author in [23, Section 3].

- Question 6.6** (a) How does the proportion of Beauville groups of order  $p^n$  that are strongly real vary as  $n$  increases?
- (b) How does the proportion of Beauville groups of order  $p^n$  that are strongly real vary as  $p$  increases?

In the opinion of the author the aforementioned work of Helleloid and Martin in [41] suggests that most Beauville 2-groups are strongly real and that for  $p$  odd very few Beauville  $p$ -groups are strongly real.

A related question is the following. One of the first questions to be asked about Beauville groups was, given a Beauville group, can we enumerate its Beauville structures? This raises the related question regarding strongly real Beauville groups.

**Question 6.7** Given a strongly real Beauville  $p$ -group, what proportion of its Beauville structures are strongly real?

### 6.5 Groups of order $p^n$ for small $n$

In [1, 25] the Beauville groups of order at most  $p^6$  are classified, however the  $p$ -groups of order at most  $p^n$  for larger values of  $n$  are known. This poses the following natural question.

**Problem 6.8** Classify the Beauville groups of order  $p^n$  for  $n > 6$ .

Since the full classification of groups of order for  $p^n$  is only known as far as  $p^9$  (with the exception of groups of order  $2^{10}$  that have also been classified — see [15, 16]) answering this question even for modest values of  $n$  is a computationally intensive task that is unlikely to be completed any time soon.

### 6.6 Beauville Spectra

The following definition was first made by Fuertes, González-Diez and Jaikin-Zapirain in [27, Definition 11].

**Definition 6.9** Let  $G$  be a finite group. The **Beauville genus spectrum** of  $G$ , denoted  $\text{Spec}(G)$ , is the set of pairs of integers  $(g_1, g_2)$  such that  $g_1 \leq g_2$  and there are curves  $C_1$  and  $C_2$  of genera  $g_1$  and  $g_2$  with the action of  $G$  on  $C_1 \times C_2$  such that  $(C_1 \times C_2)/G$  is a Beauville surface.

They went on in [27] to determine the Beauville genus spectra for the symmetric group  $S_5$  the linear group  $L_2(7)$  and abelian Beauville groups as well as showing that  $\text{Spec}(S_6) \neq \emptyset$  (though clearly this last result has been generalised by any theorem proving that another group is a Beauville group). These calculations were later pushed further to other small almost-simple groups by Pierro in his PhD thesis [50], the largest group he considered being the Matheiu group  $M_{11}$  (whose order is just 7 092) there being 87 such pairs for this group. As the orders of the groups grow the size of the Beauville genus spectrum grows too making it difficult to push these calculations for almost-simple groups much further.

If a group is a  $p$ -group, however, the story is very different. Since there is a much narrower range of element orders (being small powers of  $p$ ) and it is the orders of the elements in a Beauville structure that determine the genera of the corresponding curves, there is a much narrower range of possibilities for the Beauville spectrum making the task of finding them much more manageable, especially given that Beauville  $p$ -group tend to have low exponent. For example the following is immediate.

**Lemma 6.10** *The Beauville genus spectrum of a 2-generated  $p$ -group of exponent  $p$  is*

$$\left\{ \left( \frac{(p-1)(p-2)}{2}, \frac{(p-1)(p-2)}{2} \right) \right\}.$$

By way of example, for  $p \geq 5$  the groups  $C_p \times C_p$  and the extraspecial groups  $p_+^{1+2}$  are all groups that satisfy the hypotheses of the above lemma. This immediately raises the following interesting problem.



**Problem 6.11** Determine the Beauville genus spectrum of Beauville  $p$ -groups.

We also introduce the following.

**Definition 6.12** The **strongly real Beauville genus spectrum** of  $G$ , that we shall denote  $SRSpec(G)$  is the set of pairs of integers  $(g_1, g_2)$  such that  $g_1 \leq g_2$  and there are curves  $C_1$  and  $C_2$  of genera  $g_1$  and  $g_2$  with the action of  $G$  on  $C_1 \times C_2$  such that  $(C_1 \times C_2)/G$  is a real Beauville surface.

Since elements of larger order tend to have the property that no automorphism will map them to their inverses it seems likely to the author that  $SRSpec(G)$  will in general be much smaller than  $Spec(G)$  for most groups. In particular, if determining  $Spec(G)$  for a given group  $G$  is difficult owing to its size, the problem performing the same task for  $SRSpec(G)$  may be much more tractable.

**Problem 6.13** Determine the strongly real Beauville genus spectrum of Beauville  $p$ -groups.

As noted earlier every Beauville structure of an abelian Beauville group is necessarily strongly real so for these groups we must have  $Spec(G) = SRSpec(G)$ . For non-abelian Beauville  $p$ -groups it is likely that  $|SRSpec(G)| < |Spec(G)|$ . This motivates the following interesting question.

**Question 6.14** For a Beauville  $p$ -group  $G$  how does the size of  $SRSpec(G)$  compare to  $Spec(G)$ ? A little more specifically, how does  $|SRSpec(G)|/|Spec(G)|$  behave as  $|G| \rightarrow \infty$ ?

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